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A Black–Litterman approach to correlation stress testing

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Correlation stress testing is motivated by a well-known phenomenon: correlations change under financial crises. The adjustment of correlation matrices may be required to evaluate the potential impact of these changes. Very often, some correlations are explicitly adjusted (core correlations), with the remainder left unspecified (peripheral correlations), although it would be more natural for both core correlations and peripheral correlations to vary. However, most existing methods ignore the potential change in peripheral correlations. In this paper, we propose a Black–Litterman approach to correlation stress testing in which the stress impact on the core correlations is transmitted to the peripheral correlations through the dependence structure of the empirical correlations.

Keywords: Correlation stress testing; Scenario test; Black–Litterman; Mahalanobis distance

JEL Classification: C, G

1. Introduction

The study of dependence is of vital importance in finance. It provides a better understanding of the statistical relationships among a set of variables such as asset returns. The most popular way of measuring such dependence is by means of a correlation matrix. The most common uses of correlation matrices in finance include portfolio optimization, risk measurement and option pricing. In Markowitz’s modern portfolio theory (Markowitz 1952), a correlation matrix is used to reflect the degree of diversification among assets. In risk management, it is an important input in the calculation of value at risk (Londerstaey and Spencer 1996) under a normal or stressful environment (Kupiec 1998, Duellmann and Kick 2012). The correlation matrix is also a critical element in the valuation of investment portfolio products such as credit default swaps (Li 2000).

It is widely believed that correlations change in financial crises, and thus the adjustment of correlation matrices may be required in stress testing when forming scenarios under such stressful environments. This problem is known as correlation stress testing and has been addressed by many authors (e.g. Finger 1997, Kupiec 1998, León et al. 2002, Turkay et al. 2003, Qi and Sun 2010). The main difficulty is that changing the entries in a correlation matrix may result in an improper matrix in the sense that the resulting ‘correlation’ matrix violates the requirement of positive semidefiniteness (PSD). It is assumed that we have subjective views on the values of certain correlations and are required to adjust them to some target values. Here, correlations are divided into two groups, namely, core correlations, which are explicitly adjusted, and peripheral correlations, which are unspecified.

A common approach to correlation stress testing is the nearest correlation matrix approach, which starts with a matrix $C$ that is a guess under the given scenario when ignoring the PSD requirement. Here, we call $C$ the target matrix. When an improper $C$ is encountered, it is replaced by the proper correlation matrix nearest to $C$ based on a given distance measure. Denote the set of all proper $n \times n$ correlation matrices as $\mathcal{R}_n$. The new and proper correlation matrix $\hat{C}$ can be constructed by optimizing an objective function $f_C: \mathcal{R}_n \to \mathbb{R}$ that measures the distance between $\hat{C}$ and $C$ under an appropriate metric. Very often, $C$ is constructed by first estimating the correlation matrix econometrically and then adjusting the core correlations to the target values, which means that the peripheral correlations are kept at their original values.

Most existing methods use the Frobenius norm as the objective function (Higham 2002, Qi and Sun 2006, Grubišić and Pietersz 2007, Borsdorf and Higham 2010). Here, we call it the Frobenius correlation method. In this regard, matrix $\hat{C}$ is found by minimizing the Frobenius norm:

$$
\|\hat{C} - C\|_F = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} (\hat{c}_{ij} - c_{ij})^2}.
$$

Although the optimization problem can be easily solved, it cannot control the perturbation of each correlation.
This is desirable however, as we may have different degrees of confidence in different correlations. The situation can be tackled using the weighted version of the Frobenius norm (Pietersz and Groenen 2004), although the choice of weights is not straightforward.

The use of the Frobenius norm as the objective function may be rather arbitrary, and has no statistical meaning. Frigessi et al. (2011) proposed a so-called statistical approach, the beta correlation method, which imposes a probability distribution on the correlation matrix that is centred at $C$. Denote by $Y$ the random correlation matrix to be estimated. In this model, each correlation $y_{ij}$ is independently scaled beta distributed with support $[-1, 1]$. The variance of $y_{ij}$ is user-specified and indicates the user’s willingness to relax this entry to accommodate PSD. To lessen the computational burden that is due to the constrained optimization, Frigessi et al. (2011) consider the transformation $Y = XX^T$ and maximize the density function of $X$ with respect to the entries of $X$. However, there are several deficiencies in this model. First, the result does not maximize the density function of the correlation matrix. Maximizing the density function of $X$ with respect to the entries of $X$ and taking $Y = XX^T$ is not equivalent to maximizing the density function of $Y$ with respect to the entries of $Y$. For example, consider a beta random variable $y$ with parameters $\alpha$ and $\beta$. The value of $y$ that maximizes the density function of $y$ is equal to $(\alpha - 1)/(\alpha + \beta - 2)$. However, if we take the transformation $y = x^2$, the corresponding $y$ that maximizes the density function of $x$ is $(\alpha - 0.5)/(\alpha + \beta - 1.5)$. Second, similar to the Frobenius correlation method, the dependence structure of the correlations is unclear, and the model affords no flexibility in specifying it. As we will see in Section 3.1, results based on objective functions with different dependence structures of the correlations can be very different. An appropriate objective function should allow us to reasonably reflect the dependence structure of the correlations.

In addition to the objective function, the target matrix is also an important issue in the problem. All of the foregoing methods try to construct the target matrix by changing the core correlations to target values and keeping the peripheral correlations at their original values. However, because the target matrix represents a guess of the ‘correlation’ matrix (and may not be PSD) under the given scenario, the appropriateness of setting the peripheral correlations to their normal level when the core correlations are altered to reflect potential significant shifts is open to question. As we will see in the example in Section 3.2, the empirical correlations may be strongly correlated. If the core correlations are altered, the peripheral correlations should change accordingly. Similar arguments have been pointed out in the stress testing of risk factors. For example, Kupiec (1998) utilized the historical correlations among risk factors to predict the conditional values of peripheral factors given the stress movements in the core factors. Cherubini and Della Lunga (1999) demonstrated that the Black–Litterman model of portfolio optimization (Black and Litterman 1990, 1991, 1992) can parameterize consistent stress test scenarios. The common feature is that, when the core factors are explicitly adjusted, the peripheral factors are moved according to the covariances of the risk factors rather than assumed to be unchanged.

Unlike the nearest correlation matrix approach, which breaks down the adjustment problem into two steps, the construction and regularization of the target matrix, a unified approach to correlation stress testing is proposed in this paper. A Black–Litterman approach is used to reflect the subjective views of correlations. The stress impact on the core correlations is transmitted to the peripheral correlations through the dependence structure of the empirical correlations. A new correlation matrix is constructed by maximizing the posterior density. We show that it can also be viewed as a two-step procedure: first constructing a target matrix in a data-driven manner such that the peripheral correlations change accordingly, and then regularizing the target matrix based on a matrix norm that reasonably reflects the dependence structure of the empirical correlations.

The remainder of the paper is organized as follows. Section 2 describes a new model of correlation stress testing. Section 3 considers an international stock portfolio data-set to illustrate the use of the proposed model, and Section 4 concludes the paper.

2. The new model

For an $n \times n$ real symmetric matrix $Y$ with unit diagonal, denote by $\text{vech}[Y]$ the $N \times 1$ vector comprising all upper triangular entries of $Y$ except those on its main diagonal, where $N = n(n-1)/2$. Conversely, for an $N \times 1$ vector $y$, denote by $\text{vech}^{-1}[y]$ the $n \times n$ symmetric matrix with unit diagonal constructed from $y$.

First of all, denote by $Y$ the random correlation matrix to be estimated. Assume that $y = \text{vech}[Y]$ follows a truncated multivariate normal distribution with density

$$f(y) \propto |a \Sigma|^{	ext{−}1/2} \exp \left[-\frac{1}{2}(y - \mu)^T (a \Sigma)^{-1} (y - \mu) \right] I(y \in \mathcal{R}_n),$$

where $I\{\cdot\}$ is an indicator function; $\mathcal{R}_n$ is the set of all proper $n \times n$ correlation matrices; and $a > 0$ is the shrinkage factor that controls the perturbation of the correlations. Vector $\mu$ indicates the location of each correlation, whereas $\Sigma$ governs the dependence among the correlations. Parameters $\mu$ and $\Sigma$ can be specified according to expert opinion or estimated econometrically. In this paper, we estimate $\mu$ and $\Sigma$ by the sample mean vector and sample covariance matrix of the correlations in non-overlapping subperiods.

Suppose that there are $M$ subjective views represented by an $M \times 1$ vector $v$. Let

$$v' y \sim N_M(P y, \Omega),$$

where $P$ is a pre-specified $M \times N$ matrix in which each row identifies the correlations involved under a particular view, and $\Omega$ is the covariance matrix whose diagonal indicates the confidence of the subjective views.

By Bayes’ theorem, it can be shown that the posterior density of $y$ is

$$f(y | v) \propto |\hat{\Sigma}|^{-1/2} \exp \left[-\frac{1}{2}(y - \hat{\mu})^T \hat{\Sigma}^{-1}(y - \hat{\mu}) \right] I(y \in \mathcal{R}_n),$$

where

$$\hat{\mu} = \mu + a \Sigma P^T (a \Sigma P^T + \Omega)^{-1} (v - P \mu)$$

and

$$\hat{\Sigma} = a \left[ \Sigma - a \Sigma P^T (a \Sigma P^T + \Omega)^{-1} P \Sigma \right],$$

(1)
Following the maximum a posteriori estimation method, the new correlation matrix $\tilde{C}$ is obtained by maximizing the posterior density (1). If $\tilde{C} = \text{vech}^{-1}(\tilde{\mu}) \in \mathcal{R}_n$, the posterior density (1) is maximized at $y = \tilde{\mu}$. Otherwise, the derivation of the mode of the posterior density (1) is less straightforward, and numerical methods are required to maximize the posterior density (1).

Maximizing function (1) is equivalent to minimizing the Mahalanobis distance between $y$ and $\tilde{\mu}$ with respect to $\tilde{\Sigma}$:

$$\|y - \tilde{C}\|_{\tilde{\Sigma}}^2 = (y - \tilde{\mu})^{\top} \tilde{\Sigma}^{-1} (y - \tilde{\mu}),$$

subject to the constraint $y \in \mathcal{R}_n$, where $y = \text{vech}^{-1}(y)$ and $\tilde{C} = \text{vech}^{-1}(\tilde{\mu})$. Therefore, the proposed method can be viewed as a two-step procedure. First, we construct a target matrix $\tilde{C}$ in which the correlations change according to the covariances of the empirical correlations. Then, because $\tilde{C}$ is not guaranteed to be PSD, we further obtain the optimal proper correlation matrix $\hat{C}$ by minimizing the norm (2).

The hypersphere decomposition method proposed by Rebonato and Jäckel (2000) can be used to minimize the norm (2). Their approach turns the problem into an unconstrained optimization that can be readily solved with a standard optimization toolbox. This method parameterizes $y$ by a set of unbounded parameters based on hypersphere decomposition, and the parametrization can be improved by the triangular angle parametrization proposed by Rapisarda et al. (2007).

Note that the subjective view leads to different forms of $\mathbf{P}$. For example, if the correlation matrix is thought to have a common correlation structure, then we have

$$M = N - 1, \quad \mathbf{P} = \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -1 \end{bmatrix}$$

and

$$\mathbf{v} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}.$$

However, to simplify the discussion, we confine ourselves here to the case in which the first $M$ $y_k$’s are the core correlations, each of which is explicitly adjusted to $v_k$; that is,

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \end{bmatrix}$$

and

$$\mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_M \end{bmatrix}.$$

Moreover, because the computation involves only the ratio of the diagonal entries of $a \Sigma$ and $\Omega$, for ease of implementation, we set $a = 1$ and

$$\Omega = \frac{\tau}{1 - \tau} \mathbf{P} \Sigma \mathbf{P}^T,$$

where $0 \leq \tau \leq 1$, and $\tau = 0$ implies certainty views. For other ways of specifying $\mathbf{P}$ and $\Omega$, please refer to reviews of the Black–Litterman model such as that of Idzorek (2007).

3. Case studies

3.1. A toy example

Consider three assets in which the first two are developing country equities and the last is a developed country equity. Suppose that a finance crisis has occurred in developing countries. Increasing the (1, 2) correlation to 0.95 results in an improper correlation matrix:

$$C = \begin{bmatrix} 1 & 0.95 & 0.65 \\ 0.95 & 1 & 0.25 \\ 0.65 & 0.25 & 1 \end{bmatrix}.$$

Assume that only the (1, 3) and (2, 3) correlations are varied to allow for PSD and that the correlation between these two correlations is 0.9. Denote by $x_1$ and $x_2$ the (1, 3) and (2, 3) correlations, respectively, and $x = [x_1, x_2]$. We consider three objective functions, namely, the Frobenius norm (used in the Frobenius correlation method), scaled beta density function (used in the beta correlation method) and Mahalanobis distance (used in our method). For the scaled beta density function, set the variances as 0.01 and 0.04, respectively. For the Mahalanobis distance, we use the same means and variances for $x_1$ and $x_2$, and the correlation equals 0.9. Figure 1 compares the contour plots of the different objective functions. It illustrates that the result varies considerably depending on the objective functions used. Denote by $C_1$, $C_2$ and $C_3$ the resulting optimal correlation matrices when the respective objective functions are used.

$$C_1 = \begin{bmatrix} 1 & 0.95 & 0.59 \\ 0.95 & 1 & 0.30 \\ 0.59 & 0.30 & 1 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 1 & 0.95 & 0.64 \\ 0.95 & 1 & 0.37 \\ 0.64 & 0.37 & 1 \end{bmatrix}, \quad C_3 = \begin{bmatrix} 1 & 0.95 & 0.72 \\ 0.95 & 1 & 0.46 \\ 0.72 & 0.46 & 1 \end{bmatrix}.$$
Table 1. Sample correlation matrices for daily log returns of the 10 assets during the observation period (lower triangular) and crisis period (upper triangular).

<table>
<thead>
<tr>
<th>Asset</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Portugal</td>
<td>1</td>
<td>0.894</td>
<td>0.727</td>
<td>0.909</td>
<td>0.833</td>
<td>0.898</td>
<td>0.872</td>
<td>-0.303</td>
<td>0.552</td>
<td>0.567</td>
</tr>
<tr>
<td>Italy</td>
<td>0.448</td>
<td>1</td>
<td>0.679</td>
<td>0.935</td>
<td>0.896</td>
<td>0.963</td>
<td>0.936</td>
<td>-0.294</td>
<td>0.562</td>
<td>0.582</td>
</tr>
<tr>
<td>Greece</td>
<td>0.347</td>
<td>0.267</td>
<td>1</td>
<td>0.685</td>
<td>0.609</td>
<td>0.998</td>
<td>0.669</td>
<td>-0.230</td>
<td>0.359</td>
<td>0.378</td>
</tr>
<tr>
<td>Spain</td>
<td>0.532</td>
<td>0.643</td>
<td>0.319</td>
<td>1</td>
<td>0.843</td>
<td>0.923</td>
<td>0.883</td>
<td>-0.239</td>
<td>0.512</td>
<td>0.532</td>
</tr>
<tr>
<td>UK</td>
<td>0.427</td>
<td>0.557</td>
<td>0.271</td>
<td>0.627</td>
<td>1</td>
<td>0.945</td>
<td>0.932</td>
<td>-0.261</td>
<td>0.685</td>
<td>0.707</td>
</tr>
<tr>
<td>France</td>
<td>0.515</td>
<td>0.657</td>
<td>0.317</td>
<td>0.752</td>
<td>0.728</td>
<td>1</td>
<td>0.976</td>
<td>-0.257</td>
<td>0.576</td>
<td>0.596</td>
</tr>
<tr>
<td>Germany</td>
<td>0.529</td>
<td>0.611</td>
<td>0.332</td>
<td>0.680</td>
<td>0.632</td>
<td>0.757</td>
<td>1</td>
<td>-0.258</td>
<td>0.582</td>
<td>0.600</td>
</tr>
<tr>
<td>US Bond</td>
<td>-0.006</td>
<td>-0.033</td>
<td>-0.015</td>
<td>-0.042</td>
<td>-0.040</td>
<td>-0.036</td>
<td>-0.100</td>
<td>1</td>
<td>-0.096</td>
<td>-0.120</td>
</tr>
<tr>
<td>UK Bond</td>
<td>0.274</td>
<td>0.166</td>
<td>0.167</td>
<td>0.174</td>
<td>0.345</td>
<td>0.170</td>
<td>0.183</td>
<td>0.164</td>
<td>1</td>
<td>0.995</td>
</tr>
<tr>
<td>GBP</td>
<td>0.292</td>
<td>0.169</td>
<td>0.184</td>
<td>0.182</td>
<td>0.343</td>
<td>0.175</td>
<td>0.198</td>
<td>0.115</td>
<td>0.979</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 2. Summary statistics and estimated values of the selected $r_{ij}$’s.

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>Standard deviation</th>
<th>$y_{2,3}$</th>
<th>$y_{3,6}$</th>
<th>$y_{5,8}$</th>
<th>$y_{2,10}$</th>
<th>$y_{9,10}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y_{2,3}$</td>
<td>0.287</td>
<td>0.253</td>
<td>1</td>
<td>0.911</td>
<td>-0.520</td>
<td>0.489</td>
<td>0.409</td>
</tr>
<tr>
<td>$y_{3,6}$</td>
<td>0.327</td>
<td>0.216</td>
<td>0.911</td>
<td>1</td>
<td>-0.501</td>
<td>0.478</td>
<td>0.351</td>
</tr>
<tr>
<td>$y_{5,8}$</td>
<td>0.021</td>
<td>0.239</td>
<td>-0.520</td>
<td>-0.501</td>
<td>1</td>
<td>0.022</td>
<td>-0.309</td>
</tr>
<tr>
<td>$y_{2,10}$</td>
<td>0.176</td>
<td>0.238</td>
<td>0.489</td>
<td>0.478</td>
<td>0.022</td>
<td>1</td>
<td>0.334</td>
</tr>
<tr>
<td>$y_{9,10}$</td>
<td>0.978</td>
<td>0.020</td>
<td>0.409</td>
<td>0.351</td>
<td>-0.309</td>
<td>0.334</td>
<td>1</td>
</tr>
</tbody>
</table>

Figure 2. Scatter plot matrix of $y_{2,3}$, $y_{3,6}$, $y_{5,8}$, $y_{2,10}$ and $y_{9,10}$. 
In \( C_1 \) and \( C_2, x_1 \) is adjusted downward, whereas \( x_2 \) becomes larger. This contradicts the strongly positive correlation between \( x_1 \) and \( x_2 \). In \( C_3 \), both \( x_1 \) and \( x_2 \) become larger and \( x_2 \) deviates more because of its greater variance than \( x_1 \). Thus, the Mahalanobis distance alone reasonably reflects the dependence structure of the empirical correlations.

### 3.2. A 10-asset example

Consider a portfolio of 10 USD-denominated assets comprising the MSCI Standard Country Price Indices of Portugal, Italy, Greece, Spain, the UK, France and Germany; the USA and UK Benchmark two-year Datastream Government Bond Indices; and WM/Reuters USD/GBP Closing Spot Rate. Suppose that we perform stress testing on the portfolio against a sovereign debt crisis in Portugal, Italy, Greece and Spain. It is suspected that the correlations among the equity indices of these countries (core correlations) will increase to a high value of 0.9 but we have no idea about the other correlations (peripheral correlations).

To obtain some idea of how the correlations may change, we compare the sample correlation matrices for the daily log returns of the 10 assets during two periods, as shown in Table 1. The lower triangular part, \( C_{\text{normal}} \), gives the correlations from 2 January 1989 to 30 April 2008 (the observation period), which represents the long-run correlation matrix, and the upper triangular part, \( C_{\text{stressed}} \), gives those from 4 January 2010 to 30 June 2010 (the crisis period), during which the global financial market was influenced by the 2010 European sovereign debt crisis. We thus regard \( C_{\text{stressed}} \) as a proxy of the true correlation matrix under this scenario.

Comparing \( C_{\text{stressed}} \) with \( C_{\text{normal}} \), we see that when the core correlations increase, some of the peripheral correlations also increase, whereas others decrease. The correlations seem to be linearly correlated. To see how the empirical correlations vary over time, we compute the correlations of the grouped data. For each non-overlapping period of four months during the observation period, we compute a sample correlation matrix, for a total of 58 sample correlation matrices. If each of these matrices is regarded as an observation, we have 58 observations of \((10 \times 9)/2 = 45\) correlations. Let \( y_{ij} \) be the empirical correlation between assets \( i \) and \( j \). Figure 2 shows a scatter plot matrix of \( y_{2,3}, y_{3,6}, y_{5,8}, y_{2,10} \) and \( y_{9,10} \), and Table 2 presents their summary statistics.

<table>
<thead>
<tr>
<th>Asset</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Portugal</td>
<td>1</td>
<td>0.900</td>
<td>0.901</td>
<td>0.900</td>
<td>0.434</td>
<td>0.525</td>
<td>0.536</td>
<td>−0.006</td>
<td>0.277</td>
<td>0.290</td>
</tr>
<tr>
<td>Italy</td>
<td>0.898</td>
<td>1</td>
<td>0.898</td>
<td>0.903</td>
<td>0.541</td>
<td>0.631</td>
<td>0.598</td>
<td>−0.034</td>
<td>0.168</td>
<td>0.177</td>
</tr>
<tr>
<td>Greece</td>
<td>0.904</td>
<td>0.881</td>
<td>1</td>
<td>0.896</td>
<td>0.306</td>
<td>0.371</td>
<td>0.363</td>
<td>−0.015</td>
<td>0.163</td>
<td>0.174</td>
</tr>
<tr>
<td>Spain</td>
<td>0.896</td>
<td>0.920</td>
<td>0.861</td>
<td>1</td>
<td>0.598</td>
<td>0.699</td>
<td>0.656</td>
<td>−0.043</td>
<td>0.180</td>
<td>0.192</td>
</tr>
<tr>
<td>UK</td>
<td>0.428</td>
<td>0.553</td>
<td>0.279</td>
<td>0.620</td>
<td>1</td>
<td>0.745</td>
<td>0.642</td>
<td>−0.040</td>
<td>0.345</td>
<td>0.344</td>
</tr>
<tr>
<td>France</td>
<td>0.516</td>
<td>0.649</td>
<td>0.334</td>
<td>0.735</td>
<td>0.731</td>
<td>1</td>
<td>0.774</td>
<td>−0.036</td>
<td>0.169</td>
<td>0.173</td>
</tr>
<tr>
<td>Germany</td>
<td>0.529</td>
<td>0.608</td>
<td>0.338</td>
<td>0.674</td>
<td>0.633</td>
<td>0.760</td>
<td>1</td>
<td>−0.100</td>
<td>0.184</td>
<td>0.196</td>
</tr>
<tr>
<td>US Bond</td>
<td>−0.006</td>
<td>−0.033</td>
<td>−0.015</td>
<td>−0.042</td>
<td>−0.040</td>
<td>−0.036</td>
<td>−0.099</td>
<td>1</td>
<td>0.161</td>
<td>0.119</td>
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<td>0.171</td>
<td>0.180</td>
<td>0.186</td>
<td>0.342</td>
<td>0.173</td>
<td>0.198</td>
<td>0.115</td>
<td>0.979</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 3. \( \hat{C}_F \) (lower triangular) and \( \hat{C}_R \) (upper triangular).

<table>
<thead>
<tr>
<th>Asset</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Portugal</td>
<td>1</td>
<td>0.895</td>
<td>0.894</td>
<td>0.896</td>
<td>0.849</td>
<td>0.855</td>
<td>0.883</td>
<td>−0.090</td>
<td>0.626</td>
<td>0.661</td>
</tr>
<tr>
<td>Italy</td>
<td>0.894</td>
<td>1</td>
<td>0.893</td>
<td>0.897</td>
<td>0.936</td>
<td>0.937</td>
<td>0.963</td>
<td>−0.197</td>
<td>0.590</td>
<td>0.628</td>
</tr>
<tr>
<td>Greece</td>
<td>0.727</td>
<td>0.679</td>
<td>1</td>
<td>0.894</td>
<td>0.811</td>
<td>0.877</td>
<td>0.921</td>
<td>−0.062</td>
<td>0.512</td>
<td>0.543</td>
</tr>
<tr>
<td>Spain</td>
<td>0.909</td>
<td>0.935</td>
<td>0.685</td>
<td>1</td>
<td>0.920</td>
<td>0.952</td>
<td>0.962</td>
<td>−0.254</td>
<td>0.620</td>
<td>0.655</td>
</tr>
<tr>
<td>UK</td>
<td>0.833</td>
<td>0.896</td>
<td>0.609</td>
<td>0.843</td>
<td>1</td>
<td>0.966</td>
<td>0.950</td>
<td>−0.244</td>
<td>0.660</td>
<td>0.711</td>
</tr>
<tr>
<td>France</td>
<td>0.898</td>
<td>0.963</td>
<td>0.698</td>
<td>0.923</td>
<td>0.945</td>
<td>1</td>
<td>0.981</td>
<td>−0.268</td>
<td>0.597</td>
<td>0.640</td>
</tr>
<tr>
<td>Germany</td>
<td>0.872</td>
<td>0.936</td>
<td>0.669</td>
<td>0.883</td>
<td>0.932</td>
<td>0.976</td>
<td>1</td>
<td>−0.193</td>
<td>0.626</td>
<td>0.670</td>
</tr>
<tr>
<td>US Bond</td>
<td>−0.303</td>
<td>−0.294</td>
<td>−0.230</td>
<td>−0.239</td>
<td>−0.261</td>
<td>−0.257</td>
<td>−0.258</td>
<td>1</td>
<td>0.153</td>
<td>0.097</td>
</tr>
<tr>
<td>UK Bond</td>
<td>0.552</td>
<td>0.562</td>
<td>0.359</td>
<td>0.512</td>
<td>0.685</td>
<td>0.576</td>
<td>0.582</td>
<td>−0.096</td>
<td>1</td>
<td>0.989</td>
</tr>
<tr>
<td>GBP</td>
<td>0.567</td>
<td>0.582</td>
<td>0.378</td>
<td>0.532</td>
<td>0.707</td>
<td>0.596</td>
<td>0.600</td>
<td>−0.120</td>
<td>0.995</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 4. \( C_{\text{stressed}} \) (lower triangular) and \( \hat{C}_N \) (upper triangular).
correlated with various degrees of strength. Therefore, it is reasonable to incorporate the covariances of the correlations into correlation adjustment.

We now compare the performance of the proposed method with that of existing methods. Based on the nearest correlation matrix approach, we construct target matrix \( C \) by changing the core correlations in \( C_{\text{normal}} \) to 0.9. Because \( C \) is improper, we consider the optimal correlation matrices based on the Frobenius correlation method (denoted as \( \tilde{C}_F \)) and the beta correlation method (denoted as \( \tilde{C}_B \)). We also apply our methodology with \( \tau = 0.01 \) to adjust the correlation matrix and denote the optimal correlation matrix under the proposed method as \( \tilde{C}_N \). Tables 3 and 4 show the optimal correlation matrices under the three methods.

From table 3, we can see that the optimal matrices under the two existing methods are indeed very similar. The peripheral correlations are not adequately adjusted in either method: some need to be more positive and others more negative. From table 4, in contrast, we can observe that the optimal matrix under the proposed method is much closer to \( C_{\text{stress}} \) and \( C_{\text{normal}} \) than \( C_F \) and \( C_B \). The peripheral correlations except the \( (9, 10) \) correlation, the values of which are very similar in all three methods. It is clear that the proposed method adequately adjusts the peripheral correlations to reflect the potential changes in the stressful environment.

Following Rebonato and Jäckel (2000), we use the Frobenius norm from \( C_{\text{stress}} \) to measure the overall performance of the three methods. The Frobenius norm between \( C_{\text{normal}} \) and \( C_{\text{stress}} \) is 2.9941. That between \( C_F \) and \( C_{\text{stress}} \) is 2.7226, and that between \( C_B \) and \( C_{\text{stress}} \) is 2.7142. Hence, the beta correlation method performs only slightly better than the Frobenius correlation method, and may effect no improvement at all. The Frobenius norm between \( \tilde{C}_N \) and \( C_{\text{stress}} \) is 1.0734, which suggests that the proposed method gives a much better adjustment.

Note that there is a user-specified parameter \( \tau \in [0, 1] \) that represents the confidence of the subjective views relative to the data and that \( \tau = 0 \) implies certainty views. For the purpose of forming stress testing scenarios, \( \tau \) should be set as small as possible to ensure the core correlations are adjusted close to the targeted values. However, in practice, \( \tau \) cannot be too small so as to allow sufficient room for the correlation matrix to accommodate PSD. To investigate the proposed model’s sensitivity to \( \tau \), we try different values of \( \tau \) and compare the resultant matrix \( \tilde{C}_N(\tau) \) with \( C_{\text{stress}} \) and \( C_{\text{normal}} \). Figure 3 plots the Frobenius norm \( \| \tilde{C}_N(\tau) - C_{\text{stress}} \|_F \) and \( \| \tilde{C}_N(\tau) - C_{\text{normal}} \|_F \) against \( \tau \). We can see that \( \tilde{C}_N(\tau) \) moves closer to \( C_{\text{stress}} \) and farther from \( C_{\text{normal}} \) when a smaller \( \tau \) is used (more confident of the subjective views). However, if \( \tau \) is too small (lower than 0.2), the distance between \( \tilde{C}_N(\tau) \) and \( C_{\text{stress}} \) grows larger with a smaller \( \tau \), because a very small \( \tau \) moves the target matrix far from PSD, and thus regularizing it moves \( \tilde{C}_N(\tau) \) farther from \( C_{\text{stress}} \).

4. Conclusion

In the finance community, it is well known that in financial crises correlation matrices behave differently from the long-run correlation matrix. Correlation stress testing is necessary in which the correlation matrices are adjusted to evaluate the potential impact of changes in the correlations. Very often, some correlations are explicitly adjusted (core correlations), with others left unspecified (peripheral correlations). We consider a case in which the correlations are possibly linearly correlated with various degrees of strength. Hence, when the core correlations are altered, the peripheral correlations should vary accordingly. However, most existing methods attempt to keep the peripheral correlations as close as possible to their original values.

In this paper, a Black–Litterman approach to correlation stress testing is proposed. A new correlation matrix is constructed by maximizing the posterior density. Similar to the existing nearest correlation matrix approach, we show that the method can be viewed as a two-step procedure. We first construct a target matrix \( \tilde{C} \) in which the stress impact on the core correlations is transmitted to the peripheral correlations through the covariances of the empirical correlations. Because \( \tilde{C} \) is not guaranteed to be PSD, we further obtain the optimal proper correlation matrix \( \tilde{C} \) by minimizing the norm (2). Unlike the Frobenius norm (used in the Frobenius correlation method) and scaled beta density function (used in the beta correlation method) in which the dependence structure of the correlations is unclear and cannot be specified explicitly, the proposed Mahalanobis norm (2) is weighted by the covariances of the empirical correlations, which can be specified according to expert opinion or estimated econometrically. Empirical studies suggest that the proposed method significantly improves the results of correlation stress testing.

References

A Black–Litterman approach to correlation stress testing


